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Multi-rowed Partitions with Totally Distinct Parts

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1. INTRODUCTION

Let $t_k(n)$ be the number of k-row partitions of n (as defined in [1]), all of whose non-zero parts are distinct. For example $t_3(6) = 9$, since the partitions in question are

$$
\begin{array}{ccccccccc}
6, & 51, & 5, & 42, & 4, & 321, & 32, & 31, & 3 \\
 & & 1 & & 2 & & 1 & 2 & 2. \\
 & & & & & & & & 1
\end{array}
$$

In Section 2 we obtain an expression for the generating function $T_k(x) = \sum_{n=0}^{\infty} t_k(n)x^n$, involving a determinant whose elements are Bessel functions of imaginary argument and the Laplace transform of a previously studied series. In Section 3 we will refine our expression in the case $k = 2$.

2. GENERATING FUNCTIONS

We first consider k-row partitions π with totally distinct parts having exactly m_i non-zero parts on the ith row. (Here some of the m_i may be equal to zero.) Denote the number of such partitions by $t(n; m_1, ..., m_k)$ and let $T(x; m_1, ..., m_k) = \sum_{n=0}^{\infty} t(n; m_1, ..., m_k)x^n$.

Put $m = \sum_{i=1}^{k} m_i$. To the partition π we associate the partition π' whose

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parts are 1, 2, 3,..., m which are ordered in the same way as the parts of π. For example, if

$$\begin{array}{llll} \pi = & 12 & 9 & 6 \\ & 11 & 4 & \\ & 8 & 3 & \end{array}$$

then $m_1 = 3$, $m_2 = m_3 = 2$, $m = 7$ and

$$\begin{array}{llll} \pi' = & 7 & 5 & 3 \\ & 6 & 2 & \\ & 4 & 1 & \end{array}$$

The original partition π can be obtained from π' by superimposing an ordinary (i.e., 1-row) partition π'' of $n - \binom{m+1}{2}$ into at most m parts. For example, in the above case, $\pi'' = 5\ 5\ 4\ 4\ 3\ 2\ 2$. (Of course the largest part of π'' is to be added to the largest part of π', etc.) Conversely, if we superimpose any linear partition π'' on π' we will obtain a partition π whose associate as defined above is π'. Therefore, the partitions π associated with a given π' are generated by

$$\frac{x^{\binom{m+1}{2}}}{(1-x)(1-x^2)\cdots(1-x^m)}.$$

It follows that $T(x; m_1,..., m_k)$ is equal to this expression multiplied by the number of partitions π'. The partitions π' are the standard tableaux associated with the Young diagram of the partition $m = m_1 + m_2 + \cdots + m_k$ (cf. [3]). The number of such tableaux is well known to be

$$\frac{m! \prod_{i<j} (h_i - h_j)}{\prod_{i=1}^{k} h_i!},$$

where $h_i = m_i + k - i$. Hence we have

$$T(x; m_1,..., m_k) = \frac{m! \prod_{i<j} (h_i - h_j)}{\prod_{i=1}^{k} h_i!} \frac{x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)}$$

Summing over $m_1,..., m_k$ we obtain

$$T_k(x) = \sum_{m_1 \geqslant \cdots \geqslant m_k \geqslant 0} \frac{m! \prod_{i<j} (h_i - h_j)}{\sum_{i=1}^{k} h_i!} \frac{x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)}.$$

It is possible to replace this series by another expression which does not involve multiple summation. To do this set

$$F_k(z) = \sum_{m_1 \geqslant \cdots \geqslant m_k \geqslant 0} \frac{\prod_{i<j} (h_i - h_j)}{\prod_{i=1}^{k} h_i!} z^m$$

and

$$G_k(x, z) = \sum_{m=0}^{\infty} \frac{m!\, z^m x^{\binom{m+1}{2}}}{(1-x) \cdots (1-x^m)}.$$

By Parseval's theorem

$$T_k(x) = \frac{1}{2\pi i} \int_{|z|=1} F_k(z)\, G_k(x, z^{-1})\, dz. \tag{1}$$

As in [2] the term

$$\frac{\prod_{i<j} (h_i - h_j)}{\prod_{i=1}^{k} h_i!} z^m$$

can be written as the determinant $\det(a_{i,j})$ with $a_{i,j} = a_{i-k+h_j}$, where $a_v = z^v/v!$.

This transformation enables us to sum the series for $F_k(z)$ by applying Lemma 1 of [2]. To do this we need to compute $s = \sum_{n=0}^{\infty} a_n$ and $c_v = \sum_{n=0}^{\infty} a_n a_{n+v}$. In the present case

$$s = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

and

$$c_v = \sum_{n=0}^{\infty} \frac{z^{2n+v}}{n!(n+v)!} = I_v(2z).$$

Therefore by Lemma 1 of [2] we have for even k that $F_k(z)$ is the Pfaffian of the skew-symmetric $k \times k$ matrix

$$D_k = \begin{pmatrix} d_0 & d_1 & \cdots & d_{k-1} \\ -d_1 & d_0 & & \vdots \\ \vdots & & & \vdots \\ -d_{k-1} & \cdots & & d_0 \end{pmatrix},$$

where $d_n = I_0(2z) + 2(I_1(2z) + \cdots + I_{n-1}(2z)) + I_n(2z)$. For odd k, $F_k(z)$ is the Pfaffian of the $k+1 \times k+1$ matrix

$$D_k^1 = \begin{pmatrix} 0 & e^z & e^z & \cdots & e^z \\ -e^z & & & & \\ \vdots & & D_{k-1} & & \\ -e^z & & & & \end{pmatrix}.$$

These Pfaffians can be written as determinants using Lemma 1 of [2].

For even $k = 2l$ we obtain $F_k(z) = \det(u_{i,j})$, where

$$u_{i,j} = d_{|i-j|+1} + d_{|i-j|+3} + \cdots + d_{i+j-1} \quad (1 \leqslant i,j \leqslant l).$$

For odd $k = 2l+1$ we obtain $F_k(z) = e^z \det(v_{i,j})$, where

$$v_{i,j} = e_{|i-j|+1} + e_{|i-j|+3} + \cdots + e_{i+j-1} \quad (1 \leqslant i,j \leqslant l)$$

with

$$\begin{aligned} e_v &= -d_{v-1} + 2d_v - d_{v+1} \\ &= c_{v-1} - c_{v+1} \\ &= I_{v-1}(2z) - I_{v+1}(2z). \end{aligned}$$

For example:

$$\begin{aligned} F_1(z) &= e^z, \\ F_2(z) &= I_0(2z) + I_1(2z), \\ F_3(z) &= e^z(I_0(2z) - I_2(2z)), \end{aligned}$$

$$F_4(z) = \begin{vmatrix} I_0(2z) + I_1(2z) & I_0(2z) + 2I_1(2z) + I_2(2z) \\ I_0(2z) + 2I_1(2z) + I_2(2z) & 2I_0(2z) + 3I_1(2z) + 2I_2(2z) + I_3(2z) \end{vmatrix},$$

etc.

We turn next to the evaluation of $G_k(x, z)$. It is convenient first to express $G_k(x, z)$ as the Laplace transform of another function $H_k(x, w)$ (regarded as a function of z). Put

$$H_k(x, w) = \sum_{m=0}^{\infty} \frac{w^m x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)}.$$

It is easy to see from the ratio test that for any fixed x with $|x| < 1$, $H_k(x, w)$ is an entire function of w, and $|H_k(w, w)| < e^{c|w|}$, where c is independent of w (but depends on x). The Laplace integral $\mathscr{L}(H_k(x, w))$ converges in some half plane; moreover, this Laplace integral can be interchanged with the summation over m so that

$$\begin{aligned}\mathscr{L}(H_k(x, w)) &= \int_0^\infty \sum_{m=0}^\infty \frac{w^m x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)} e^{-wz}\,dw \\ &= \sum_{m=0}^\infty \frac{x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)} \int_0^\infty w^m e^{-wz}\,dw \\ &= \sum_{m=0}^\infty \frac{x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)} \frac{m!}{z^{m+1}} \\ &= G_k(x, z^{-1})/z.\end{aligned}$$

It is easy to see from the ratio test that $G_k(x, z)$ is an entire function of z for fixed x with $|x| < 1$. Thus we can write $G_k(x, z^{-1}) = z\ \mathscr{L}(H_k(x, w))$, where the right-hand side is to be thought of as continued analytically beyond the region of convergence of the Laplace integral.

It is well known (cf. [3]) that $H_k(x, w) = \prod_{v=1}^\infty (1 - wx^v)^{-1}$. Therefore we obtain the formula

$$T_k(x) = \frac{1}{2\pi i} \int_{|z|=1} F_k(z)\,\mathscr{L}\left(\prod_{v=1}^\infty (1 - wx^v)^{-1}\right) z\,dz.$$

3. Further Discussion of $k = 2$

From Eq. (1) we have

$$\begin{aligned}T_2(x) &= \frac{1}{2\pi i} \int_{|z|=1} (I_0(2z) + I_1(2z))\,G_2(x, z^{-1})\,dz \\ &= \sum_{v=0}^\infty \left[\frac{(2v)!}{v!^2} \frac{x^{\binom{2v+1}{2}}}{(1-x)\cdots(1-x^{2v})}\right. \\ &\quad \left. + \frac{(2v+1)!}{v!(v+1)!} \frac{x^{\binom{2v+1}{2}}}{(1-x)\cdots(1-x^{2v+1})}\right] \\ &= \sum_{m=0}^\infty \binom{m}{[m/2]} \frac{x^{\binom{m+1}{2}}}{(1-x)\cdots(1-x^m)}.\end{aligned}$$

This can be written as

$$\frac{1}{2\pi i} \int_{|z|=1} K_2(cz)\,H_2(x_1(cz)^{-1})\,dz,$$

where $H(x, z)$ is as defined above and

$$K_2(z) = \sum_{m=0}^{\infty} \binom{m}{[m/2]} z^m \tag{2}$$

and c is any non-zero number inside the circle of convergence of (2). An easy calculation shows that

$$K_2(z) = \frac{2z + 1 - (1 - 4z^2)^{1/2}}{2z(1 - 4z^2)^{1/2}}$$

so that the radius of convergence of (2) is $\frac{1}{2}$. Therefore we have

$$T_2(x) = \frac{1}{2\pi i} \int_{|z|=1} \frac{2cz + 1 - (1 - 4c^2z^2)^{1/2}}{2cz\ 1 - (4c^2z^2)^{1/2}} \prod_{v=1}^{\infty} (1 - c^{-1}z^{-1}x^v)^{-1}\, dz,$$

where c is any non-zero complex number with $0 < |c| < \frac{1}{2}$. This curious formula for $T_2(x)$ could be made the basis of the study of the asymptotic behavior of $t_2(n)$ as $n \to \infty$. We will, however, use a different approach to be described in a subsequent paper.

REFERENCES

1. B. GORDON AND L. HOUTEN, Notes on plane partitions, I, *J. Combinatorial Theory* **4** (1968), 72–80.
2. B. GORDON AND L. HOUTEN, Notes on plane partitions, II, *J. Combinatorial Theory* **4** (1968), 81–99.
3. P. A. MACMAHON, "Combinatory Analysis," Cambridge Univ. Press, Cambridge, 1916.